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Quantum Deformation of $igl(n)$ Algebra on Quantum Space

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Abstract

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We study quantum deformed $gl(n)$ and $igl(n)$ algebras on a quantum space discussing multi-parametric extension. We realize elements of deformed $gl(n)$ and $igl(n)$ algebras by a quantum fermionic space. We investigate a map between deformed $igl(2)$ algebras of our basis and other basis.

1. Introduction

Quantum deformation of groups and algebras has been studied in recent years [1-4]. In addition to deformation of Lie groups and algebras, some types of inhomogeneous groups and algebras are deformed, e.g., deformations of $igl(n)$, $iso(n)$ and Poincaré algebras [5-15].

Quantum spaces are non-commutative spaces to represent quantum groups and their commutation relations between coordinates and derivatives are governed by the R-matrices for corresponding quantum groups [16-18]. Further, one of quantum deformed $su(2)$ algebras is constructed based on the two-dimensional quantum space in ref.[19]. In ref.[20] supersymmetric extension of the quantum space, i.e., quantum bosonic and fermionic spaces, is discussed. The quantum $su(2)$ is extended into construction of quantum deformed $su(m|n)$ algebra on the quantum superspace in ref.[15], which also shows that generators of the deformed $su(2)$ algebra can be realized in terms of the two-dimensional quantum fermionic coordinates and derivatives. In this paper, we hence generalize these construction of quantum algebras into an inhomogeneous case. The purpose of this paper is to explain how to construct quantum deformed algebra $igl(n)$ on fermionic quantum space as well as on bosonic one.

We employ the quantum Grassmann representation to construct quantum algebras. This method is used in some publications mentioned above. Grassmannian construction of quantum groups has a powerful advantage because its representation becomes very simple compared to usual construction of quantum groups. However, it is not clear whether or not the deformed algebras based on this method possess non-trivial Hopf algebra structure. None of the non-trivial structure has been found so far. It is hence important to find a connection of our approach to known Hopf algebra structures. In this sense, we should discuss also a relation between our deformed algebra and

other deformed algebra which exhibits non-trivial Hopf algebra structure.

This paper is organized as follows. In section 2, we explain how to construct the deformed $su(n)$ algebras on the quantum spaces in preliminary. As new results, we mention a construction of the $su(2)$ algebra on quantum space with two deformation parameters and show a realization of deformed $su(n)$ in terms of fermionic coordinates and derivatives. In section 3, we define a dilatation generator D in order to construct deformed $gl(n)$ and $igl(n)$ algebras. In section 4, we exhibit a relation of our deformed $igl(2)$ algebra to Castellani's one. Section 5 is devoted to conclusion and discussion.

2. Deformed $su(n)$ algebra on quantum space

Quantum space is the non-commutative space representing a quantum group. Differential calculi on bosonic and fermionic quantum spaces for the quantum deformed $SU(n)$ are discussed in refs.[18, 20]. The quantum group $GL_q(n)$ has more deformation parameters than $SU_q(n)$ does. For example $GL_q(2)$ has two deformation parameters. Quantum space with multi-parameters were studied in ref. [21, 22, 20].

First of all, we show a deformed $su(2)$ algebra on 2-dimensional quantum fermionic space with two deformation parameters as a simple example of multi-parameter cases. Bosonic case is remarked at the end of this section. We set up commutation relations between θ^α and ∂_α with two parameters, r and q , as follows,

$$\begin{aligned} q\theta^1\theta^2 &= -\theta^2\theta^1, & \partial_1\partial_2 &= -\frac{r^2}{q}\partial_2\partial_1, \\ q\partial_2\theta^1 &= -\theta^1\partial_2, & \partial_1\theta^2 &= -\frac{q}{r^2}\theta^2\partial_1, \\ \partial_\alpha\theta_\alpha &= 1 - \theta_\alpha\partial_\alpha + (r^{-2} - 1) \sum_{\beta>\alpha} \theta^\beta\partial_\beta, \end{aligned} \tag{2.1}$$

where $(\theta^\alpha)^2 = (\partial_\alpha)^2 = 0$. When we take $q = r$, we obtain the quantum plane with one parameter.

Commutation relations among generators are found as follows. First we consistently define operations of generators on the above quantum space. We then construct commutation relations among generators with the aid of the operations on the quantum space. Now let us consider two generators denoted by T_2^1 and T_1^2 , which correspond to $\theta^1\partial_2$ and $\theta^2\partial_1$ in the classical limit ($r, q \rightarrow 1$). Following ref.[19], we assume that T_2^1 acts on θ^α and ∂_α as follows,

$$\begin{aligned} T_2^1\theta^\alpha &= a_\alpha\theta^\alpha T_2^1 + \delta_2^\alpha\theta^2, \\ T_2^1\partial_\alpha &= b_\alpha\partial_\alpha + c\delta_\alpha^1\partial_2. \end{aligned} \quad (2.2)$$

We investigate consistency between (2.1) and (2.2). For example, we calculate $T_2^1(\theta^2)^2$ as

$$T_2^1(\theta^2)^2 = (a_2)^2(\theta^2)^2 T_2^1 + a_2\theta^2\theta^1 + \theta^1\theta^2. \quad (2.3)$$

Compared with (2.1), we get $a_2 = 1/q$. Other relations are obtained in the same way

$$\begin{aligned} [T_2^1, \theta^1]_{r^2/q} &= 0, \quad [T_2^1, \theta^2]_{1/q} = \theta^1, \\ [T_2^1, \partial_1]_{q/r^2} &= -\frac{q}{r^2}\partial_2, \quad [T_2^1, \partial_2]_q = 0, \end{aligned} \quad (2.4)$$

where $[A, B]_h \equiv AB - hBA$. Similarly for T_1^2 , its operations on the quantum space are

$$\begin{aligned} [T_1^2, \theta^1]_q &= \theta^2, \quad [T_1^2, \theta^2]_{q/r^2} = 0, \\ [T_1^2, \partial_1]_{1/q} &= 0, \quad [T_1^2, \partial_2]_{r^2/q} = -\frac{r^2}{q}\partial_1. \end{aligned} \quad (2.5)$$

As for a Cartan element H_1 , it is defined by a commutation relation between T_2^1 and T_1^2 . Since we have the following relations on the coordinates

$$[T_1^2 T_2^1, \theta^1]_{r^2} = \frac{r^2}{q}\theta^2 T_2^1, \quad [T_2^1 T_1^2, \theta^1]_{r^2} = q^{-1}\theta^2 T_2^1 + \theta^1, \quad (2.6)$$

it seems natural to define H_1 as

$$H_1 \equiv r^{-1}[T_1^2, T_2^1]_{r^2}, \quad (2.7)$$

in order to eliminate the linear terms of T_2^1 on the right hand side of (2.6). Accordingly, operations of H_1 on θ^α and ∂_α become

$$\begin{aligned} [H_1, \theta^1]_{r^2} &= -r\theta^1, & [H_1, \partial_1]_{1/r^2} &= r^{-1}\partial_1, \\ [H_1, \theta^2]_{1/r^2} &= r^{-1}\theta^2, & [H_1, \partial_2]_{r^2} &= -r\partial_2. \end{aligned} \quad (2.8)$$

Using (2.4), (2.5) and (2.8), we have the following relations,

$$[H_1, T_2^1]_{r^4} = -r^2(r + r^{-1})T_2^1, \quad [H_1, T_1^2]_{1/r^4} = r^{-2}(r + r^{-1})T_1^2, \quad (2.9)$$

Eqs.(2.7) and (2.9) compose a deformed $su(2)$ algebra as an abstract algebra. Note that this deformed $su(2)$ algebra on the quantum plane with the two parameters include only one parameter r and the operations of H_1 is independent on q . The operations of only T_β^α , (2.4) and (2.5), include the two parameters. For one parameter deformation, it was shown in ref.[15] that the deformed $su(2)$ generators are realized as

$$\begin{aligned} T_2^1 &= \theta^1\partial_2, & T_1^2 &= \theta^2\partial_1 \\ H_1 &= r^{-1}\theta^2\partial_2 - r\theta^1\partial_1. \end{aligned} \quad (2.10)$$

This is of course true even in the case with two deformation parameters.

In a way similar to the above derivation, we can construct a deformed $su(n)$ algebra using n -dimensional quantum space. We here show the results of a case with only one deformation parameter r , for simplicity. Multi-parametric extensions can be derived in a straightforward way. Commutation relations between fermionic coordinates θ^α ($\alpha = 1 \sim n$) and their derivatives ∂_α are obtained as follows,

$$r\theta^\alpha\theta^\beta = -\theta^\beta\theta^\alpha, \quad \partial_\alpha\partial_\beta = -r\partial_\beta\partial_\alpha, \quad (\alpha \leq \beta),$$

$$\begin{aligned}
r\partial_\alpha\theta^\beta &= -\theta^\beta\partial_\alpha, \quad (\alpha \neq \beta), \\
\partial_\alpha\theta_\alpha &= 1 - \theta_\alpha\partial_\alpha + (r^{-2} - 1) \sum_{\beta>\alpha} \theta^\beta\partial_\beta.
\end{aligned} \tag{2.11}$$

For all generators $T_{\alpha+1}^\alpha$, $T_\alpha^{\alpha+1}$ and H_α of deformed $su(2)$ parts, operations on the above quantum space θ^α and ∂_α are cast into the following commutation relations

$$\begin{aligned}
[T_{\alpha+1}^\alpha, \theta^\alpha]_r &= 0, \quad [T_{\alpha+1}^\alpha, \partial_\alpha]_{1/r} = -r^{-1}\partial_{\alpha+1}, \\
[T_{\alpha+1}^\alpha, \theta^{\alpha+1}]_{1/r} &= \theta^\alpha, \quad [T_{\alpha+1}^\alpha, \partial_{\alpha+1}]_r = 0, \\
[T_\alpha^{\alpha+1}, \theta^\alpha]_r &= \theta^{\alpha+1}, \quad [T_\alpha^{\alpha+1}, \partial_\alpha]_{1/r} = 0, \\
[T_\alpha^{\alpha+1}, \theta^{\alpha+1}]_{1/r} &= 0, \quad [T_\alpha^{\alpha+1}, \partial_{\alpha+1}]_r = -r\partial_\alpha, \\
[H_\alpha, \theta^\alpha]_{r^2} &= -r\theta^\alpha, \quad [H_\alpha, \partial_\alpha]_{1/r^2} = r^{-1}\partial_\alpha, \\
[H_\alpha, \theta^{\alpha+1}]_{1/r^2} &= r^{-1}\theta^{\alpha+1}, \quad [H_\alpha, \partial_{\alpha+1}]_{r^2} = -r\partial_{\alpha+1}.
\end{aligned} \tag{2.12}$$

These generators commute with the other coordinates and derivatives.

Hence, a deformed $su(n)$ algebra as an abstract algebraic relation is obtained as follows: for the $su(2)$ parts,

$$\begin{aligned}
[H_\alpha, T_{\alpha+1}^\alpha]_{r^4} &= -r^2(r + r^{-1})T_{\alpha+1}^\alpha, \quad [H_\alpha, T_\alpha^{\alpha+1}]_{1/r^4} = r^{-2}(r + r^{-1})T_\alpha^{\alpha+1}, \\
[T_\alpha^{\alpha+1}, T_{\alpha+1}^\alpha]_{r^2} &= rH_\alpha,
\end{aligned} \tag{2.13}$$

and the other generators T_γ^α of the deformed $su(n)$ are as [15]

$$T_\gamma^\alpha \equiv [T_\beta^\alpha, T_\gamma^\beta]_r, \quad (\alpha < \beta < \gamma \text{ or } \alpha > \beta > \gamma). \tag{2.14}$$

Using (2.12), we can obtain operation of T_γ^α on the quantum space. Also the whole algebra of the deformed $su(n)$ can be obtained [15].

In closing the section, we show a realization of all the generators $T_{\alpha+1}^\alpha$, $T_\alpha^{\alpha+1}$ and H_α which satisfy the commutation relations (2.13) and (2.14) using

the above n -dimensional quantum space as below. Differently from two-dimensional case, it is somewhat complicated rather than (2.10). Here we define Θ^α , $\Theta^{\alpha+1}$, D_α and $D_{\alpha+1}$ as

$$\begin{aligned}\Theta^\alpha &\equiv \mu'_\alpha \mu_\alpha^{-1/2} \theta^\alpha, & D_\alpha &\equiv \mu'_\alpha \mu_\alpha^{-1/2} \partial_\alpha, \\ \Theta^{\alpha+1} &\equiv \mu_{\alpha+1}^{-1/2} \theta^{\alpha+1}, & D_{\alpha+1} &\equiv \mu_{\alpha+1}^{-1/2} \partial_{\alpha+1},\end{aligned}\tag{2.15}$$

where μ'_α and μ_α are defined as [23, 15]

$$\mu'_\alpha \equiv [1 + (r^{-2} - 1) \theta^{\alpha+1} \partial_{\alpha+1}]^{1/2}, \quad \mu_\alpha \equiv 1 + (r^{-2} - 1) \sum_{\beta > \alpha} \theta^\beta \partial_\beta. \tag{2.16}$$

It is easy to show that these coordinates and derivatives, Θ^α , $\Theta^{\alpha+1}$, D_α and $D_{\alpha+1}$, satisfy the same commutation relations as the two-dimensional quantum space. Using them, we can realize the generators, $T_{\alpha+1}^\alpha$, $T_\alpha^{\alpha+1}$ and H_α as

$$\begin{aligned}T_{\alpha+1}^\alpha &= \Theta^\alpha D_{\alpha+1}, & T_\alpha^{\alpha+1} &= \Theta^{\alpha+1} D_\alpha \\ H_\alpha &= r^{-1} \Theta^{\alpha+1} D_{\alpha+1} - r \Theta^\alpha D_\alpha.\end{aligned}\tag{2.17}$$

In the case of quantum bosonic spaces, we can construct the same deformed $su(n)$ algebra in the same way as explained above. The operation rule of the generators on the bosonic quantum space is same as (2.12). However, we can not realize the deformed $su(n)$ algebra in a simple way as (2.10) or (2.17) using coordinates and derivatives of the quantum bosonic space.

3. Quantum deformed $gl(n)$ and $igl(n)$ algebras

In this section, we define a dilatation generator D in order to construct quantum deformed $gl(n)$ and $igl(n)$ algebras. Let us start with the case of the 2-dimensional quantum fermionic space. As a first step, we utilize

$\theta^1\partial_1$ and $\theta^2\partial_2$ in order to realize a generator D which decouples to the three generators of (2.10). Namely, we define the generator D as

$$D \equiv r^{-1}(\theta^1\partial_1 + \theta^2\partial_2). \quad (3.1)$$

We see that the above generator operates on the quantum space as

$$[D, \theta^\alpha]_{1/r^2} = r^{-1}\theta^\alpha, \quad [D, \partial_\alpha]_{r^2} = -r\partial_\alpha, \quad (3.2)$$

where $\alpha = 1, 2$ and thus we can show that D commutes with the three generators of (2.10), even in the case with the two deformation parameters. We therefore understand that the deformed $gl(2)$ algebra is obtained in terms of T_2^1, T_1^2, H_1 and D . Even though we have considered the quantum space with two deformation parameters, this deformed $gl(2)$ algebra depends only on one parameter r . Another deformation parameter appears when we consider an inhomogeneous extension. In addition to these generators, ∂_1, ∂_2 define a deformed $igl(2)$ algebra. The second deformation parameter q appears in the commutation relations between T_β^α and ∂_α as well as the commutation relation between ∂_1 and ∂_2 .

In general, we can construct the deformed $gl(n)$ and $igl(n)$ algebras similarly. Here we define the dilatation generator D as follows,

$$D \equiv r^{-1} \sum_{\alpha=1}^n \theta^\alpha \partial_\alpha. \quad (3.3)$$

The generator D commutes with all the generators of the deformed $su(n)$ algebra, T_γ^α and H_α . These generators span the deformed $gl(n)$ algebra. Moreover D operates on the quantum space as

$$[D, \theta^\alpha]_{1/r^2} = r^{-1}\theta^\alpha, \quad [D, \partial_\alpha]_{r^2} = -r\partial_\alpha. \quad (3.4)$$

We can construct the deformed $igl(n)$ algebra in terms of $T_\gamma^\alpha, H_\alpha, D$ and ∂_α . In other words, all the operators $\theta^\alpha\partial_\gamma$ of the n -dimensional quantum space construct the deformed $gl(n)$ algebra. In addition to these, the derivatives

∂_α derive the deformed $igl(n)$ algebra. It is easy to extend the construction of $igl(n)$ to cases with multi-parameters.

Finally, we comment on the construction on the quantum bosonic space. In this case, we can define D which has the same commutation relations as (3.4). We can construct the deformed $igl(n)$ algebra in terms of T_j^i , H_i , D and $\partial/\partial x_i$, although these deformed $gl(n)$ generators can not be realized by x^i and $\partial/\partial x_i$ as (2.10).

4. Map to Castellani's deformation

We here present some mapping relations from our deformation (for simplicity, we analyze one parameter case substituting r with q) to another deformation which possesses a non-trivial Hopf algebra structure discussed by Castellani [13]. There is a difference between gl and igl cases. To show it, we first consider a map to the following algebraic relation for deformed $gl(2)$ (hereafter we call it $\hat{\chi}$ -algebra),

$$\begin{aligned}\hat{\chi}_+\hat{\chi}_1 - r^{-2}\hat{\chi}_1\hat{\chi}_+ &= \hat{\chi}_+, & \hat{\chi}_1\hat{\chi}_- - r^{-2}\hat{\chi}_-\hat{\chi}_1 &= \hat{\chi}_-, \\ r^2\hat{\chi}_+\hat{\chi}_- - r^{-2}\hat{\chi}_-\hat{\chi}_+ &= \hat{\chi}_2 - \hat{\chi}_1, & \hat{\chi}_\pm X - r^{\pm 2}X\hat{\chi}_\pm &= 0, \\ [\hat{\chi}_1, \hat{\chi}_2] &= 0, & X &= (1 - r^2)X\hat{\chi}_2 - r.\end{aligned}\tag{4.1}$$

The deformed $gl(2)$ of Castellani is obtained through the rescaling $\chi_i = X\hat{\chi}_i$ ($i = 1, 2, +, -$). We introduce the following recombination for convenience,

$$A = \frac{q}{q + q^{-1}}(qD + q^{-1}H), \quad B = \frac{1}{q + q^{-1}}(D - H),\tag{4.2}$$

and consider the case of $\hat{\chi}_+ = T_1^2$ and $\hat{\chi}_- = T_2^1$. Expanding χ_1, χ_2 and X as linear combinations of our basis A, B and AB , we find a family of solutions which satisfy the $\hat{\chi}$ -algebra (4.1) as follows

$$\hat{\chi}_2 = aA + bB + cAB,$$

$$\hat{\chi}_1 = (a - r^2)A + (b + r^{-2})B + (c + r^2q^2 - r^{-2})AB, \quad (4.3a)$$

$$X = (x + r)A + (y + r)B + (z - r - y)AB - r,$$

where the parameters a, b, c, x, y, z are arbitrary as long as they satisfy

$$b = \frac{a - 1}{r^2}, \quad x = \frac{r}{1 - (1 - r^2)a}, \quad y = \frac{r^3}{1 - (1 - r^2)a},$$

$$\frac{z - r - y}{1 - r^2} = z(a + q^{-2}(b + c)) + (x - y)b + xc. \quad (4.3b)$$

If we fix the parameter a , the degree of freedom of undetermined parameters is two because b, x, y are also fixed in conformity with the value of a . This means that we have a two-parameter family of relations between different deformation parameters q and r of each deformed $gl(2)$. Similarly, we can observe another family of mapping for the choice $\chi_+ \propto T_2^1$ and $\chi_- \propto T_1^2$. Note that the above case is $\chi_+ = xT_1^2$ and $\chi_- = yT_2^1$. For example, choosing $\chi_+ = r^2T_2^1$ and $\chi_- = r^2T_1^2$, we obtain

$$\chi_1 = r^3B + \alpha AB, \quad \chi_2 = rA + \beta AB, \quad (4.4a)$$

with

$$(1 - r^2) \left(r\beta + (r + \beta q^{-2})(\beta - \alpha - r^3) \right) + r(\alpha - \beta) = r^4(q^2 - 1). \quad (4.4b)$$

Although we have fixed some coefficients in the second family (4.4), we still have unknown parameters α and β . Namely, as for isomorphisms on deformed $gl(2)$ from our deformation to Castellani's one, we have a relation between deformation parameters which is expressed by the curves with two parameters as (4.3b) or (4.4b).

Next, we consider isomorphisms on inhomogeneous extensional part. In this case, all the redundant free parameters are fixed and the relation between deformation parameters become a straight-line without free parameter. Following Castellani's deformation, we have two deformation parameters r and s in their algebra,

$$r^2\chi_1P_1 - P_1\chi_1 = (1 - r^2)P_2\chi_- - rP_1,$$

$$\begin{aligned}
r^2\chi_2P_2 - P_2\chi_2 &= -rP_2, \\
s\chi_+P_1 - P_1\chi_+ &= r^2(r^2 - 1)\chi_2P_2 - r^3P_2, \\
r^2\chi_-P_2 - sP_2\chi_- &= s(r^2 - 1)\chi_2P_1 - srP_1, \\
s\chi_+P_2 - P_2\chi_+ &= 0, \quad r^2\chi_-P_1 - sP_1\chi_- = 0, \\
[\chi_2, P_1] &= 0, \quad [\chi_1, P_2] = (1 - r^2)\frac{s}{r^2}\chi_+P_1, \\
r^2P_1P_2 - sP_1P_2 &= 0.
\end{aligned} \tag{4.5}$$

Under this extension, there is no solution for arbitrary free parameters in both cases of (4.3) and (4.4). We have to fix the values of free parameters in order to satisfy (4.5). This seems to uniquely determine the values of free parameters, however in addition, we must not forget considering a map between our ∂_i and P_i . Then we can find a solution which satisfies (4.5). We have no solution for the first map (4.3), while the second type (4.4) has the solution

$$\alpha = \beta = 0, \tag{4.6}$$

and the equation (4.4b) reduces into

$$r = q. \tag{4.7}$$

Mapping from our coordinate derivatives to P_1, P_2 and another deformation parameter s in (4.5) are given by

$$P_1 = \partial_1 + (r^2 - 1)A\partial_1, \quad P_2 = r^{-1}\partial_2, \tag{4.8}$$

$$s = q^{-1}. \tag{4.9}$$

Hence, the equations (4.4) with (4.6)-(4.9) connect our deformed $igl(2)$ to (4.1) and (4.5).

As concluding remarks, we notice a duality on $gl(2)$ part of the map (4.4) with the case of (4.6). Exchanging $1 \leftrightarrow 2$, $+$ \leftrightarrow $-$ and $q \leftrightarrow q^{-1}$, we can transform (4.4) into the following another map

$$\chi_+ = r^2T_1^2, \quad \chi_- = r^2T_2^1, \tag{4.10a}$$

$$\chi_1 = r^3 A, \quad \chi_2 = r B, \quad \text{with} \quad q = 1/r. \quad (4.10b)$$

As mentioned above, there is no free parameters in this case and so the relation between q and r is fixed. It should be noticed that this map is not included in (4.3) as a special case.

5. Conclusion and discussion

In this paper, we have introduced a method how to construct a deformed $igl(n)$ algebra based on the fermionic and/or bosonic quantum space. In section 2, we have discussed the construction of a deformed $sl(n)$ algebra which is independent of realizations. After that, incorporating the dilatation and translation operators on the quantum space, we have embodied a deformed $igl(n)$ algebra in section 3. In the case of $igl(2)$ deformation, we have shown a mapping relation from our deformation to other one [13].

As a result of the analysis of section 4, there exist at least two categories of isomorphisms between our deformed $gl(2)$ to the Castellani deformation. The first one, to which (4.3) belongs, does not hold in the case of the inhomogeneous extension. The second one, to which (4.4) belongs, includes a mapping relation which holds in the inhomogeneous case. As for (4.10), we have not examined whether its inhomogeneous case holds or not, however it might belong to the second one for the reasons that there exists a duality to the (4.4) with (4.6) and that it certainly different from the type of (4.3) as previously suggested. It would be interesting to investigate complete classification of these kind of maps and a duality relation.

The advantage of our approach consists in simple construction of representations of quantum algebras. Owing to this property, we have found the relation between our deformation and the other which possesses non-trivial Hopf algebra structure. It is not so trivial whether or not we can find such map relation in other cases. It is also interesting to investigate our approach

applying to the $iso(n)$, $isp(n)$, Poincaré and super algebras.

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